RECOLLEMENT OF ADDITIVE QUOTIENT CATEGORIES

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Abstract. In this note, we define a recollement of additive categories, and prove that such a recollement can induce a recollement of their quotient categories. As an application, we get a recollement of quotient triangulated categories induced by mutation pairs.

1. Introduction

The recollement of triangulated categories was introduced in a geometric setting by Beilinson, Bernstein, and Deligne [1], and has been studied in an algebraic setting by Cline, Parshall and Scott [3, 4]. The recollement of abelian category was formulated by Franjou and Pirashvili [5]. The recollment of categories plays an important role in algebraic geometry and in representation theory.

Quotient categories give a way to produce abelian categories [10, 12]. Such as, Koenig and Zhu showed that the quotient category $\mathcal{C}/\mathcal{T}$ is abelian provided that $\mathcal{T}$ is a cluster tilting subcategory of a triangulated category $\mathcal{C}$ [10]. A recollement of triangulated categories can induced a recollement of quotient abelian categories [2, 9]. On the other hand, quotient categories also give a way to produce triangulated categories [6, 7]. Such as, Iyama and Yoshino proved that if $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair and $\mathcal{Z}$ is extension-closed, then $\mathcal{Z}/\mathcal{D}$ is a triangulated category [7]. It is natural to ask that whether a recollement of categories can induce a recollement of quotient triangulated categories.

In this note, to unify the recollment of abelian categories and the recollement of triangulated categories, we define the notion of recollement of addtive categories. In section 2, we prove that a recollement of additive categories can induce a recollement of their subcategories and a recollement of their quotient categories under certain conditions. In section 3, we apply the main result of section 2 to triangulated categories and obtain a recollement of quotient triangulated categories induced by mutation pairs.

2. Recollement of additive categories

Throughout this note, all subcategories of a category are full and closed under isomorphisms, direct sums and direct summands. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. We denote by $\mathrm{Im}F$ the subcategory of $\mathcal{B}$ generated by objects $F(X)$ with $X \in \mathcal{A}$, and by $\mathrm{Ker}F$ the subcategory of $\mathcal{A}$ generated by objects X with $F(X) = 0$. Let $f : X \to Y$, $g : Y \to Z$ be two morphisms in $\mathcal{A}$, and M

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an object in $\mathcal{C}$, we denote by gf the composition of f and g, by f_* the morphism $\mathrm{Hom}_{\mathcal{A}}(M, f) : \mathrm{Hom}_{\mathcal{A}}(M, X) \to \mathrm{Hom}_{\mathcal{A}}(M, Y)$, and by f^* the morphism $\mathrm{Hom}_{\mathcal{A}}(f, M) : \mathrm{Hom}_{\mathcal{A}}(Y, M) \to \mathrm{Hom}_{\mathcal{A}}(X, M)$.

Definition 2.1. Let $\mathcal{A}, \mathcal{A}'$ and $\mathcal{A}''$ be additive categories. The diagram

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "i_*"] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{A}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.1}$$

of additive functors is a *recollement of additive category $\mathcal{A}$ relative to additive categories $\mathcal{A}'$ and $\mathcal{A}''$*, if the following conditions are satisfied:

(R1) $(i^*, i_*), (i_*, i^!), (j_!, j^*)$ and (j^*, j_*) are adjoint pairs;

(R2) $i_*, j_!$ and j_* are full embeddings;

(R3) $\mathrm{Im} i_* = \mathrm{Ker} j^*$.

Remark 2.2. (1) If all involved categories are abelian, then the diagram (2.1) is a recollement of abelian categories. If all involved categories are triangulated, and all involved functors are exact, then by [13, Theorem 3.2], the diagram (2.1) is a recollement of triangulated categories in the sense of Beilinson-Bernstein-Deligne.

(2) For any adjoint pair (F, G), it is well-known that F is fully faithful if and only if the unit $\varepsilon : id \to GF$ is a natural equivalence. If F is a full embedding, then there exists an adjoint pair (F, G') such that the unit $\varepsilon : id \to GF$ is the identity. Thus if necessary, we may assume that $i^* i_* = id, i^! i_* = id$ and $j^* j_* = id$ in Definition 2.1.

Let $\mathcal{A}$ be an additive category and $\mathcal{X}$ a subcategory. Then by definition the quotient category $\mathcal{A}/\mathcal{X}$ has the same objects as $\mathcal{A}$, and the set of morphisms from A to B in the quotient category is defined as the quotient group $\mathrm{Hom}_{\mathcal{A}}(A, B)/[\mathcal{X}](A, B)$, where $[\mathcal{X}](A, B)$ is the subgroup of morphism in $\mathcal{A}$ factoring through some object in $\mathcal{X}$. For $f : A \to B$ a morphism in $\mathcal{A}$, we denote by $\overline{f}$ its residue class in the quotient category.

The following lemma is well-known.

Lemma 2.3. *Let $\mathcal{A}$ be an additive category and $\mathcal{X}$ a subcategory. There exists an additive functor $Q_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}/\mathcal{X}$ such that*

(1) $Q_{\mathcal{A}}(X)$ is null for every $X \in \mathcal{X}$;

(2) For any additive category $\mathcal{B}$ and additive functor $F : \mathcal{A} \to \mathcal{B}$ such that $F(X)$ is null for every $X \in \mathcal{X}$, there exists a unique additive functor $\widetilde{F} : \mathcal{A}/\mathcal{X} \to \mathcal{B}$ such that $F = \widetilde{F} \cdot Q_{\mathcal{A}}$, i.e., we have the following commutative diagram of functors.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathcal{B} \\ \mathcal{A}/\mathcal{X} \arrow[ur, dotted, "\widetilde{F}"'] & \end{tikzcd}$$

The following lemma is basic, but it is important in our proof. For convenience, we give a proof.

Lemma 2.4. *Let $\mathcal{A}$ and $\mathcal{A}'$ be additive categories, $i^* : \mathcal{A} \to \mathcal{A}'$ and $i_* : \mathcal{A}' \to \mathcal{A}$ be additive functors. If $\mathcal{X}$ is a subcategory of $\mathcal{A}$ and $\mathcal{X}'$ a subcategories of $\mathcal{A}'$, satisfying $i^*(\mathcal{X}) \subset \mathcal{X}'$ and $i_*(\mathcal{X}') \subset \mathcal{X}$, then*

(1) The functor i^* induces an additive functor $\widetilde{i^*} : \mathcal{A}/\mathcal{X} \to \mathcal{A}'/\mathcal{X}'$;

(2) The functor i_* induces an additive functor $\widetilde{i_*} : \mathcal{A}'/\mathcal{X}' \to \mathcal{A}/\mathcal{X}$;
(3) If (i^*, i_*) is an adjoint pair, then so is $(\widetilde{i^*}, \widetilde{i_*})$.

Proof. (1)Since $Q_{\mathcal{A}'} i^*(X)$ is null for any $X \in \mathcal{X}$, by Lemma 2.3 there exists a unique additive functor $\widetilde{i^*} : \mathcal{A}/\mathcal{X} \to \mathcal{A}'/\mathcal{X}'$ such that the following diagram is commutative

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "i^*"] \arrow[d, "Q_{\mathcal{A}}"'] & & \mathcal{A}' \arrow[d, "Q_{\mathcal{A}'}"] \\ \mathcal{A}/\mathcal{X} \arrow[rr, dotted, "\widetilde{i^*}"] & & \mathcal{A}'/\mathcal{X}'. \end{tikzcd}$$

We can prove (2) similarly.

(3) Note that (i^*, i_*) is an adjoint pair , there exist two natural isomorphisms

$$\eta_{A,A'} : \mathrm{Hom}_{\mathcal{A}'}(i^*(A), A') \to \mathrm{Hom}_{\mathcal{A}}(A, i_*(A')),$$

$$\tau_{A,A'} : \mathrm{Hom}_{\mathcal{A}}(A, i_*(A')) \to \mathrm{Hom}_{\mathcal{A}'}(i^*(A), A')$$

such that $\tau_{A,A'}\eta_{A,A'} = 1$ and $\eta_{A,A'}\tau_{A,A'} = 1$, where $A \in \mathcal{A}$, $A' \in \mathcal{A}'$. We claim that $\eta_{A,A'}$ induces a natural isomorphism

$$\widetilde{\eta}_{A,A'} : \mathrm{Hom}_{\mathcal{A}'/\mathcal{X}'}(\widetilde{i^*}(A), A') \to \mathrm{Hom}_{\mathcal{A}/\mathcal{X}}(A, \widetilde{i_*}(A')).$$

We first define a map $\widetilde{\eta}_{A,A'} : \mathrm{Hom}_{\mathcal{A}'/\mathcal{X}'}(\widetilde{i^*}(A), A') \to \mathrm{Hom}_{\mathcal{A}/\mathcal{X}}(A, \widetilde{i_*}(A'))$. For any $\overline{f} \in \mathrm{Hom}_{\mathcal{A}'/\mathcal{X}'}(\widetilde{i^*}(A), A')$, define $\widetilde{\eta}_{A,A'}(\bar{f}) := \overline{\eta_{A,A'}(f)}$. In order to show that the definition is reasonable, it suffices to prove that $\overline{\eta_{A,A'}(f)}$ does not depend on the choice of f. Let $\overline{f_1} = \overline{f_2} \in \mathrm{Hom}_{\mathcal{A}'/\mathcal{X}'}(\widetilde{i^*}(A), A')$, i.e., there exist two morphisms $s : i^*(A) \to X$ and $t : X' \to A'$ such that the following diagram is commutative

$$\begin{tikzcd} i^*(A) \arrow[rr, "f_1 - f_2"] \arrow[dr, "s"'] & & A' \\ & X' \arrow[ur, "t"'] & \end{tikzcd}$$

where $X' \in \mathcal{X}'$. It is easy to see that $\eta_{A,A'}(f_1) - \eta_{A,A'}(f_2) = \eta_{A,A'}(f_1 - f_2) = i_*(f_1 - f_2)\varepsilon_A = i_*(ts)\varepsilon_A = i_*(t)i_*(s)\varepsilon_A$, where ε_A is the adjunction morphism. Hence the following diagram is commutative

$$\begin{tikzcd} A \arrow[rr, "\eta_{A,A'}(f_1) - \eta_{A,A'}(f_2)"] \arrow[dr, "i_*(s)\varepsilon_A"'] & & i_*(A') \\ & i_*(X') \arrow[ur, "i_*(t)"'] & \end{tikzcd}$$

Since $i_*(\mathcal{X}') \subset \mathcal{X}$, we have $i_*(X') \in \mathcal{X}$, thus $\overline{\eta_{A,A'}(f_1)} = \overline{\eta_{A,A'}(f_2)}$. Therefore $\widetilde{\eta}_{A,A'}$ is a morphism between $\mathrm{Hom}_{\mathcal{A}'/\mathcal{X}'}(\widetilde{i^*}(A), A')$ and $\mathrm{Hom}_{\mathcal{A}/\mathcal{X}}(A, \widetilde{j_*}(A'))$.

Similarly, we can define a morphism

$$\widetilde{\tau}_{A,A'} : \mathrm{Hom}_{\mathcal{A}/\mathcal{X}}(A, \widetilde{i_*}(A')) \to \mathrm{Hom}_{\mathcal{A}'/\mathcal{X}'}(\widetilde{i^*}(A), A')$$

by $\widetilde{\tau}_{A,A'}(g) := \overline{\tau_{A,A'}(g)}$, where $\overline{g} \in \mathrm{Hom}_{\mathcal{A}/\mathcal{X}}(A, \widetilde{i_*}(A'))$.

Note that $\tau_{A,A'}\eta_{A,A'} = 1$ and $\eta_{A,A'}\tau_{A,A'} = 1$, we obtain that $\tilde{\tau}_{A,A'}\tilde{\eta}_{A,A'} = 1$ and $\tilde{\eta}_{A,A'}\tilde{\tau}_{A,A'} = 1$. Therefore $\tilde{\eta}_{A,A'}$ is a bijection.

It remains to prove that $\widetilde{\eta}_{A,A'}$ is a natural transformation. For any morphism $\overline{h} \in \mathrm{Hom}_{\mathcal{A}/\mathcal{X}}(B, A)$, we claim that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}'/\mathcal{X}'}(\widetilde{i^*}(A), A') \arrow[r, "\widetilde{\eta}_{A,A'}"] \arrow[d, "(\widetilde{i^*}\bar{h})^*"'] & \mathrm{Hom}_{\mathcal{A}/\mathcal{X}}(A, \widetilde{i_*}(A')) \arrow[d, "\bar{h}^*"] \\
\mathrm{Hom}_{\mathcal{A}'/\mathcal{X}'}(\widetilde{i^*}(B), A') \arrow[r, "\widetilde{\eta}_{B,A'}"] & \mathrm{Hom}_{\mathcal{A}/\mathcal{X}}(B, \widetilde{i_*}(A'))
\end{tikzcd}$$

In fact, for any morphism $\bar{f} \in \mathrm{Hom}_{\mathcal{A}'/\mathcal{X}'}(\widetilde{i^*}(A), A')$, since $\eta_{A,A'}$ is a natural transformation, we have $\eta_{A,A'}(f) \cdot h = \eta_{B,A'}(f \cdot i^* h)$. Thus

$$\widetilde{\eta}_{A,A'}(\bar{f}) \cdot \bar{h} = \overline{\eta_{A,A'}(f) \cdot h} = \overline{\eta_{B,A'}(f \cdot i^*(h))} = \widetilde{\eta}_{B,A'}\overline{(f \cdot i^*(h))} = \widetilde{\eta}_{B,A'}(\bar{f} \cdot \widetilde{i^*}(\overline{h})),$$

that is, $\widetilde{\eta}_{B,A'} \cdot (\widetilde{i^*}\bar{h})^* = \bar{h}^* \cdot \widetilde{\eta}_{A,A'}$. Hence $\widetilde{\eta}_{A,A'}$ is natural in the first variable.

Using similar arguments as before one can show that $\widetilde{\eta}_{A,A'}$ is natural in the second variable. This finishes the proof. $\square$

Now we can state and prove our main theorem in this section.

Theorem 2.5. *Let*

$$\begin{tikzcd}
\mathcal{A}' \arrow[r, "i_*"] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{A}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

be a recollement of additive categories, $\mathcal{X}$ an additive subcategory of $\mathcal{A}$ such that $i_*i^*(\mathcal{X}) \subset \mathcal{X}$, $j_*j^*(\mathcal{X}) \subset \mathcal{X}$, $i_*i^!(\mathcal{X}) \subset \mathcal{X}$ and $j_!j^*(\mathcal{X}) \subset \mathcal{X}$. Then

(1) the diagram

$$\begin{tikzcd}
i^*(\mathcal{X}) \arrow[r, "\bar{i}_*"] & \mathcal{X} \arrow[l, bend right, "\bar{i}^*"'] \arrow[l, bend left, "\bar{i}^!"] \arrow[r, "\bar{j}^*"] & j^*(\mathcal{X}) \arrow[l, bend right, "\bar{j}_!"'] \arrow[l, bend left, "\bar{j}_*"]
\end{tikzcd}$$

is a recollement of additive categories, where $\bar{i}_*, \bar{i}^!, \bar{i}^*, \bar{j}^*, \bar{j}_*$ and $\bar{j}_!$ are restriction functors of $i_*, i^!, i^*, j^*, j_*$ and $j_!$ respectively;

(2) there exists a diagram of additive functors

$$\begin{tikzcd}
\mathcal{A}'/i^*(\mathcal{X}) \arrow[r, "\tilde{i}_*"] & \mathcal{A}/\mathcal{X} \arrow[l, bend right, "\tilde{i}^*"'] \arrow[l, bend left, "\tilde{i}^!"] \arrow[r, "\tilde{j}^*"] & \mathcal{A}''/j^*(\mathcal{X}) \arrow[l, bend right, "\tilde{j}_!"'] \arrow[l, bend left, "\tilde{j}_*"]
\end{tikzcd} \qquad (2.2)$$

Moreover, the diagram (2.2) is a recollement of additive categories if and only if $\mathcal{X} \subset Kerj^*$.

Proof. (1) (R1) and (R2) are trivial. For (R3), it is clear that $\bar{j}^*\bar{i}_* = 0$, hence $\mathrm{Im}\bar{i}_* \subset \mathrm{Ker}\bar{j}^*$. On the other hand, since $\mathrm{Ker}\bar{j}^* = \mathrm{Ker}j^* \bigcap \mathcal{X} = \mathrm{Im}\ i_* \bigcap \mathcal{X}$, for any $X \in \mathrm{Ker}\bar{j}^*$, there exists $A' \in \mathcal{A}'$ such that $X = i_*(A')$. It follows that $X = i_*(A') = i_*i^*i_*(A') = i_*i^*(X) \in \mathrm{Im}\bar{i}_*$. This shows that $\mathrm{Im}\bar{i}_* = \mathrm{Ker}\bar{j}^*$.

(2) By Lemma 2.4, the six functors $i^*, i_*, i^!, j_!, j^*, j_*$ induce six additive functors $\tilde{i}^*, \tilde{i}_*, \tilde{i}^!, \tilde{j}_!, \tilde{j}^*, \tilde{j}_*$, respectively. Furthermore, $(\tilde{i}^*, \tilde{i}_*)$, $(\tilde{i}_*, \tilde{i}^!)$, $(\tilde{j}_!, \tilde{j}^*)$ and $(\tilde{j}^*, \tilde{j}_*)$ are adjoint pairs. Since i_* is a full embedding, we may assume that $i^!i_* = id_{\mathcal{A}'}$. Thus $\tilde{i}^!\tilde{i}_* = id_{\mathcal{A}'/i^*(X)}$. It follows that $\tilde{i}_*$ is a full embedding. Similarly, $\tilde{j}_!, \tilde{j}_*$ are full embeddings. Since $j^*i_* = 0$, we have $\widetilde{j^*}\widetilde{i_*} = 0$, which implies that $\mathrm{Im}\widetilde{i_*} \subset \mathrm{Ker}\widetilde{j^*}$. To end the proof, it remains to prove that $\mathrm{Ker}\widetilde{j^*} = \mathrm{Im}\widetilde{i_*}$ if and only if $\mathcal{X} \subset \mathrm{Ker}j^*$.

In fact, if $\mathcal{X} \subset \mathrm{Ker} j^*$, we have $\mathrm{Ker}\widetilde{j}^* = j^{*-1}(j^*(\mathcal{X}))/\mathcal{X} = \mathrm{Ker} j^*/\mathcal{X} = \mathrm{Im} i_*/\mathcal{X} = \mathrm{Im}\widetilde{i}_*$. On the other hand, assume that $\mathrm{Im}\widetilde{i}_* = \mathrm{Ker}\widetilde{j}^*$. By the definition of quotient category and that of quotient functor, $\mathrm{Im}\widetilde{i}_*$ and $\mathrm{Im} i_*$ have the same object, and the objects of $\mathcal{X}$ is a subclass of the objects of $\mathrm{Ker}\widetilde{j}^*$. Since $\mathrm{Im} i_* = \mathrm{Ker} j^*$, the objects of $\mathcal{X}$ is a subclass of the objects of $\mathrm{Ker} j^*$. Therefore, $\mathcal{X} \subset \mathrm{Ker} j^*$.

This finishes the proof. □

Corollary 2.6. *Let*

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "i_*"] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{A}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

be a recollement of additive categories. Let $\mathcal{X}'$ be a subcategory of $\mathcal{A}'$ and $\mathcal{X}''$ a subcategory of $\mathcal{A}''$, satisfying $i^*j_*(\mathcal{X}'') \subset \mathcal{X}'$ and $i^!j_!(\mathcal{X}'') \subset \mathcal{X}'$. If $\mathcal{X} = \{X \in \mathcal{A} | j^*(X) \in \mathcal{X}'', i^*(X) \in \mathcal{X}', i^!(X) \in \mathcal{X}'\}$, Then

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "\overline{i}_*"] & \mathcal{X} \arrow[l, bend right, "\overline{i}^*"'] \arrow[l, bend left, "\overline{i}^!"] \arrow[r, "\overline{j}^*"] & \mathcal{X}'' \arrow[l, bend right, "\overline{j}_!"'] \arrow[l, bend left, "\overline{j}_*"] \end{tikzcd}$$

is a recollement of additive categories, where $\overline{i}_*, \overline{i}^!, \overline{i}^*, \overline{j}^*, \overline{j}_*$ and $\overline{j}_!$ are the restriction functors of $i_*, i^!, i^*, j^*, j_*$ and $j_!$, respectively.

Proof. For every $X' \in \mathcal{X}'$, since $j^*i_*(X') = 0 \in \mathcal{X}''$, $i^*i_*(X') = X' \in \mathcal{X}'$, and $i^!i_*(X') = X' \in \mathcal{X}'$, we obtain that $i_*(\mathcal{X}') \subset \mathcal{X}$. Thus $\mathcal{X}' = i^*i_*(\mathcal{X}') \subset i^*(\mathcal{X})$. On the other hand, by definition of $\mathcal{X}$, $i^*(\mathcal{X}) \subset \mathcal{X}'$, so $i^*(\mathcal{X}) = \mathcal{X}'$. Similarly, we can show that $j_*(\mathcal{X}'') \subset \mathcal{X}$ and $j^*(\mathcal{X}) = \mathcal{X}''$.

Now we have $i_*i^*(\mathcal{X}) = i_*(\mathcal{X}') \subset \mathcal{X}$, $j_*j^*(\mathcal{X}) = j_*(\mathcal{X}'') \subset \mathcal{X}$ and $i_*i^!(\mathcal{X}) \subset i_*(\mathcal{X}') \subset \mathcal{X}$. It is sufficient to show that $j_!j^*(\mathcal{X}) \subset \mathcal{X}$ by Theorem 2.5(1). It is easy to check that $j_!(\mathcal{X}'') \subset \mathcal{X}$. Therefore, $j_!j^*(\mathcal{X}) = j_!(\mathcal{X}'') \subset \mathcal{X}$. This finishes the proof. □

Corollary 2.7. *Let*

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "i_*"] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{A}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

be a recollement of additive categories, $\mathcal{X}'$ a subcategories of $\mathcal{A}'$. Then

$$\begin{tikzcd} \mathcal{A}'/\mathcal{X}' \arrow[r, "\widetilde{i}_*"] & \mathcal{A}/i_*(\mathcal{X}') \arrow[l, bend right, "\widetilde{i}^*"'] \arrow[l, bend left, "\widetilde{i}^!"] \arrow[r, "\widetilde{j}^*"] & \mathcal{A}'' \arrow[l, bend right, "\widetilde{j}_!"'] \arrow[l, bend left, "\widetilde{j}_*"] \end{tikzcd}$$

is a recollement of additive categories.

Proof. If we take $\mathcal{X} = i_*(\mathcal{X}')$, then $j^*\mathcal{X} = 0$, $i_*i^*(\mathcal{X}) = i_*i^!(\mathcal{X}) = \mathcal{X}$ and $j_*j^*(\mathcal{X}) = j_!j^*(\mathcal{X}) = 0$. The result immediately follows from Theorem 2.5(2). □

3. Recollement of triangulated quotient categories induced by mutation pairs

Let $\mathcal{D}$ be a subcategory of an additive category $\mathcal{C}$. A morphism $f : A \to B$ in $\mathcal{C}$ is called $\mathcal{D}$-*epic*, if for any $D \in \mathcal{D}$, the sequence $\mathrm{Hom}_{\mathcal{C}}(D, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(D, B) \longrightarrow 0$ is exact. A *right $\mathcal{D}$-approximation* of X in $\mathcal{C}$ is a $\mathcal{D}$-epic map $f : D \to X$, with $D \in \mathcal{D}$. The subcategory $\mathcal{D}$ is said to be a *contravariantly finite* if any object

A of $\mathcal{C}$ has a right $\mathcal{X}$-approximation. We can defined $\mathcal{D}$-*monic* morphism, *left* $\mathcal{D}$-*approximation* and *covariantly finite subcategory* dually. The subcategory $\mathcal{D}$ is called *functorially finite* if it is both contravariantly finite and covariantly finite.

Definition 3.1. (cf.[7],[11]) Let $\mathcal{C}$ be a triangulated category, and $\mathcal{D} \subseteq \mathcal{Z}$ be subcategories of $\mathcal{C}$. Then the pair $(\mathcal{Z}, \mathcal{Z})$ is called a $\mathcal{D}$-*mutation pair* if the following conditions are satisfied.

(1) For any object $X \in \mathcal{Z}$, there exists a triangle

$$X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} TX$$

where $Y \in \mathcal{Z}$, $D \in \mathcal{D}$, f is a left $\mathcal{D}$-approximation and g is a right $\mathcal{D}$-approximation.

(2) For any object $Y \in \mathcal{Z}$, there exists a triangle

$$X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} TX$$

where $X \in \mathcal{Z}$, $D \in \mathcal{D}$, f is a left $\mathcal{D}$-approximation and g is a right $\mathcal{D}$-approximation.

Lemma 3.2. *Let $\mathcal{C}$, $\mathcal{C}'$ be triangulated categories, and $F : \mathcal{C} \to \mathcal{C}'$ a full and exact functor. If $\mathcal{D} \subset \mathcal{Z}$ are subcategories of $\mathcal{C}$, and $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair, then $(F\mathcal{Z}, F\mathcal{Z})$ is a $F\mathcal{D}$-mutation pair.*

Proof. For any object $FX \in F\mathcal{Z}$, where $X \in \mathcal{Z}$, there exists a triangle in $\mathcal{C}$

$$X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} TX$$

where $Y \in \mathcal{Z}$, $D \in \mathcal{D}$, f is a left $\mathcal{D}$-approximation and g is a right $\mathcal{D}$-approximation. Since F is exact, we have a triangle in $\mathcal{C}'$

$$FX \xrightarrow{Ff} FD \xrightarrow{Fg} FY \longrightarrow TFX.$$

Let $u' \in \mathrm{Hom}_{\mathcal{C}'}(FX, FD')$. Since F is full, there is a morphism $u \in \mathrm{Hom}_{\mathcal{C}}(X, D')$ such that $Fu = u'$. Since f is a left $\mathcal{D}$-approximation, there is a morphism $d \in \mathrm{Hom}_{\mathcal{C}}(D, D')$, such that $u = df$. Hence $u' = Fu = FdFf$. Then Ff is a left $F\mathcal{D}$-approximation. Similarly, Fg is a right $F\mathcal{D}$-approximation. This shows that $(F\mathcal{Z}, F\mathcal{Z})$ satisfies condition (1) of Definition 3.1. Similarly, we can show that it also satisfies condition (2) of Definition 3.1. Therefore $(F\mathcal{Z}, F\mathcal{Z})$ is a $F\mathcal{D}$-mutation pair. This finishes the proof. $\square$

Let $(\mathcal{Z}, \mathcal{Z})$ be a $\mathcal{D}$-mutation pair. For any object $X \in \mathcal{Z}$, fix a triangle

$$X \xrightarrow{\alpha_X} D_X \xrightarrow{\beta_X} M \xrightarrow{\gamma_X} TX$$

where $M \in \mathcal{Z}$, $D_X \in \mathcal{D}$, α_X is a left $\mathcal{D}$-approximation and β_X is a right $\mathcal{D}$-approximation. We can define an equivalent functor $\sigma : \mathcal{Z}/\mathcal{D} \to \mathcal{Z}/\mathcal{D}$ such that $\sigma X = M$. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ be a triangle in $\mathcal{C}$ with $X, Y, Z \in \mathcal{Z}$, where f is $\mathcal{D}$-monic. Then there exists a commutative diagram where rows are triangles.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "y"] & Z \arrow[r, "h"] \arrow[d, "z"] & TX \arrow[d, equal] \\
X \arrow[r, "\alpha_X"] & D_X \arrow[r, "\beta_X"] & \sigma X \arrow[r, "\gamma_X"] & TX
\end{tikzcd}$$

We have the following sextuple in the quotient category $\mathcal{Z}/\mathcal{D}$

$$(*) \qquad X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} Z \xrightarrow{\overline{z}} \sigma X$$

We define Φ the class of triangles in $\mathcal{Z}/\mathcal{D}$ as the sextuples which are isomorphic to $(*)$. With notation as above, we have the following lemma.

Lemma 3.3. *([7]) Let $(\mathcal{Z}, \mathcal{Z})$ be a $\mathcal{D}$-mutation pair in a triangulated category $\mathcal{C}$. If $\mathcal{Z}$ is extension-closed, then $(\mathcal{Z}/\mathcal{D}, \sigma, \Phi)$ is a triangulated category.*

Now assume that $(\mathcal{C}, \mathcal{C})$ is a $\mathcal{D}$-mutation pair in a triangulated category $\mathcal{C}$. Then $\mathcal{C}/\mathcal{D}$ is a triangulated category by Lemma 3.3. Let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor between triangulated categories. If F is full and dense, then $(\mathcal{C}', \mathcal{C}')$ is a $F\mathcal{D}$-mutation pair by Lemma 3.2, thus $\mathcal{C}'/F\mathcal{D}$ is also a triangulated category by Lemma 3.3. It is easy to see that F induces an additive functor $\widetilde{F} : \mathcal{C}/\mathcal{D} \to \mathcal{C}'/F\mathcal{D}$ satisfying the following commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "Q_{\mathcal{C}}"'] & \mathcal{C}' \arrow[d, "Q_{\mathcal{C}'}"] \\
\mathcal{C}/\mathcal{D} \arrow[r, "\widetilde{F}"] & \mathcal{C}'/F\mathcal{D}.
\end{tikzcd}$$

Moreover, we have the following lemma.

Lemma 3.4. *The functor $\widetilde{F} : \mathcal{C}/\mathcal{D} \to \mathcal{C}'/F\mathcal{D}$ is exact.*

Proof. Let $X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} Z \xrightarrow{\overline{z}} \sigma X$ be a standard triangle in $\mathcal{C}/\mathcal{D}$, then there exists the following commutative diagram of triangles in $\mathcal{C}$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "y"] & Z \arrow[r, "h"] \arrow[d, "z"] & TX \arrow[d, equal] \\
X \arrow[r, "\alpha_X"] & D_X \arrow[r, "\beta_X"] & \sigma X \arrow[r, "\gamma_X"] & TX
\end{tikzcd}$$

Thus we have the following commutative diagram of triangles in $\mathcal{C}'$

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, equal] & FY \arrow[r, "Fg"] \arrow[d, "Fy"] & FZ \arrow[r, "\phi_X \cdot Fh"] \arrow[d, "Fz"] & TFX \arrow[d, equal] \\
FX \arrow[r, "F(\alpha_X)"] & FD_X \arrow[r, "F(\beta_X)"] & F\sigma X \arrow[r, "\phi_X \cdot F(\gamma_X)"] & TFX
\end{tikzcd}$$

where $\phi_X : FTX \to TFX$ is a natural equivalence. Note that $F(\alpha_X)$ is a left $F\mathcal{D}$-approximation and $F(\beta_X)$ is a right $F\mathcal{D}$-approximation, thus $\sigma FX = F\sigma X$. By definition we obtain that $FX \xrightarrow{\overline{Ff}} FY \xrightarrow{\overline{Fg}} FZ \xrightarrow{\overline{Fz}} \sigma(FX)$ is a triangle in $\mathcal{C}'/F\mathcal{D}$. Namely, $\widetilde{F}X \xrightarrow{\widetilde{F}\overline{f}} \widetilde{F}Y \xrightarrow{\widetilde{F}\overline{g}} \widetilde{F}Z \xrightarrow{\widetilde{F}\overline{z}} \sigma(\widetilde{F}X)$ is a triangle in $\mathcal{C}'/F\mathcal{D}$. Therefore $\widetilde{F}$: $\mathcal{C}/\mathcal{D} \to \mathcal{C}'/F\mathcal{D}$ is an exact functor. $\square$

Theorem 3.5. *Let*

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{C}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

be a recollement of triangulated categories, $\mathcal{D} \subset Kerj^*$ *be subcategories of* $\mathcal{C}$ *and* $(\mathcal{C}, \mathcal{C})$ *be a* $\mathcal{D}$*-mutation pair. Then*

$$\begin{tikzcd} \mathcal{C}'/i^*(\mathcal{D}) \arrow[r, "\widetilde{i_*}"] & \mathcal{C}/\mathcal{D} \arrow[l, shift right=2.5ex, "\widetilde{i^*}"'] \arrow[l, shift left=2.5ex, "\widetilde{i^!}"] \arrow[r, "\widetilde{j^*}"] & \mathcal{C}'' \arrow[l, shift right=2.5ex, "\widetilde{j_!}"'] \arrow[l, shift left=2.5ex, "\widetilde{j_*}"] \end{tikzcd}$$

is a recollement of triangulated categories.

Proof. Since $\mathcal{D} \subset \mathrm{Ker}j^* = \mathrm{Im}i_*$, there exists $\mathcal{D}'$ of subcategory of $\mathcal{C}'$ such that $i_*(\mathcal{D}') = \mathcal{D}$. Thus $i_*i^*(\mathcal{D}) = i_*i^*i_*(\mathcal{D}') = i_*(\mathcal{D}') = \mathcal{D}$. By Corollary 2.7, the following diagram

$$\begin{tikzcd} \mathcal{C}'/i^*(\mathcal{D}) \arrow[r, "\widetilde{i_*}"] & \mathcal{C}/\mathcal{D} \arrow[l, shift right=2.5ex, "\widetilde{i^*}"'] \arrow[l, shift left=2.5ex, "\widetilde{i^!}"] \arrow[r, "\widetilde{j^*}"] & \mathcal{C}'' \arrow[l, shift right=2.5ex, "\widetilde{j_!}"'] \arrow[l, shift left=2.5ex, "\widetilde{j_*}"] \end{tikzcd}$$

is a recollement of additive categories.

Since $(\mathcal{C}, \mathcal{C})$ is a $\mathcal{D}$-mutation pair, $\mathcal{C}/\mathcal{D}$ is a triangulated category. Since the functors i^* and j^* are full and dense, $\mathcal{C}'/i^*(\mathcal{D})$ is a triangulated category by Lemma 3.2 and $\widetilde{j^*}, \widetilde{i^*}$ are exact functors by Lemma 3.4. Since the left and right adjoint functors of an exact functor are exact functors too, $\widetilde{i_*}, \widetilde{i^!}, \widetilde{j_!}, \widetilde{j^*}$ are exact functors. This finishes the proof by Remark 2.2(1). □

From now on, assume that k is an algebraically closed field, and all categories are k-linear categories with finite dimensional Hom spaces and split idempotents. Let $\mathcal{C}$ be a triangulated category with a Serre functor S. Then $\mathcal{C}$ has AR-triangles and we denote by τ the Auslander-Reiten translation functor. If $\mathcal{X}$ is a functorially finite subcategory of $\mathcal{C}$ such that $\tau\mathcal{X} = \mathcal{X}$, then $(\mathcal{C}, \mathcal{C})$ is a $\mathcal{X}$-mutation pair by [8, Lemma 2.2]. Thus $\mathcal{C}/\mathcal{X}$ is a triangulated category. According to Theorem 2.5, we have the following corollary.

Corollary 3.6. *Let*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "i_*"] & \mathcal{C} \arrow[l, shift right=2.5ex, "i^*"'] \arrow[l, shift left=2.5ex, "i^!"] \arrow[r, "j^*"] & \mathcal{C}'' \arrow[l, shift right=2.5ex, "j_!"'] \arrow[l, shift left=2.5ex, "j_*"] \end{tikzcd}$$

be a recollement of triangulated categories, $\mathcal{X}$ *a functorially finite subcategory of* $\mathcal{C}$ *such that* $\tau\mathcal{X} = \mathcal{X}$ *and* $\mathcal{X} \subset Kerj^*$*, then*

$$\begin{tikzcd} \mathcal{C}'/i^*(\mathcal{X}) \arrow[r, "\widetilde{i_*}"] & \mathcal{C}/\mathcal{X} \arrow[l, shift right=2.5ex, "\widetilde{i^*}"'] \arrow[l, shift left=2.5ex, "\widetilde{i^!}"] \arrow[r, "\widetilde{j^*}"] & \mathcal{C}'' \arrow[l, shift right=2.5ex, "\widetilde{j_!}"'] \arrow[l, shift left=2.5ex, "\widetilde{j_*}"] \end{tikzcd}$$

is a recollement of triangulated categories.

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